

# Immersed solutions of Plateau's problem for piecewise smooth boundary curves with small total curvature

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## Abstract

We provide a new proof of the classical result that any closed rectifiable Jordan curve  $\Gamma \subset \mathbb{R}^3$  being piecewise of class  $C^2$  bounds at least one *immersed* minimal surface of disc-type, under the additional assumption that the total curvature of  $\Gamma$  is smaller than  $6\pi$ . In contrast to the methods due to Osserman [10], Gulliver [6] and Alt [1], [2], our proof relies on a polygonal approximation technique, using the existence of immersed solutions of Plateau's problem for polygonal boundary curves, provided by the first author's accomplishment [3] of Garnier's ideas in [5].

## 1 Main result and introduction

Given a closed rectifiable Jordan curve  $\Gamma$ , the classical solution to Plateau's problem obtained by Douglas [4] and Radó [12] in the early 1930's yields the existence of a *generalized* minimal surface of disc-type spanning  $\Gamma$ : its interior is an immersed surface, except possibly at a finite number of branch points. It has been a famous and long out-standing problem whether in fact such branch points actually occur. Entirely immersed solutions to Plateau's problem were finally achieved in the 1970's by Osserman [10], Gulliver [6] and Alt [1], [2]. In this paper we provide a new proof of this now classical result:

**Theorem 1.** *Any closed rectifiable Jordan curve  $\Gamma \subset \mathbb{R}^3$  being piecewise of class  $C^2$  and with total curvature smaller than  $6\pi$  bounds at least one immersed minimal surface of disc-type.*

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Here a minimal surface of disc-type is a (non-constant) map  $X : \bar{B} \rightarrow \mathbb{R}^3$ , defined on the closure of the unit disc  $B = \{w = (u, v) \in \mathbb{R}^2 \mid |w| < 1\}$ , of class  $C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$  which is harmonic and conformally parametrized on  $B$ , i.e. which meets

$$\Delta X = 0 \quad \text{and} \quad |X_u|^2 - |X_v|^2 = 0 = \langle X_u, X_v \rangle \quad \text{on } B.$$

It is termed *immersed* if it additionally satisfies  $|X_u| > 0$  on  $B$ , i.e. if it does not have any branch point on  $B$  (see Definition 2). Moreover for any given closed Jordan curve  $\Gamma \subset \mathbb{R}^3$  we shall abbreviate by  $\mathcal{C}(\Gamma)$  the set of those surfaces  $X \in C^0(\bar{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$  which are *bounded* by  $\Gamma$ , thus whose restrictions  $X|_{\partial B}$  map  $\partial B$  continuously and weakly monotonically onto  $\Gamma$  with mapping degree 1. Finally a closed Jordan curve  $\Gamma \subset \mathbb{R}^3$  is termed “piecewise of class  $C^2$ ” if there is a homeomorphic parametrization  $\gamma : [0, 2\pi) \xrightarrow{\cong} \Gamma$  of  $\Gamma$  which is twice continuously differentiable in every point of  $[0, 2\pi)$  with the exception of at most finitely many points  $0 \leq t_1 < \dots < t_m < 2\pi$  in which there still exist the one-sided derivatives  $\dot{\gamma}(t_i+) := \lim_{t \searrow t_i} \frac{\gamma(t) - \gamma(t_i)}{t - t_i}$  and  $\dot{\gamma}(t_i-) := \lim_{t \nearrow t_i} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}$ .

The central tool of our proof of Theorem 1 is the following existence result for Plateau’s problem in the case of polygonal boundary curves, which could finally be achieved by the first author in [3] (see the Main Theorem on p. 9), by accomplishing Garnier’s examination [5] of second-order Fuchsian differential equations defined on the Riemann sphere  $\mathbb{C} \cup \{\infty\}$ : the method relies on the resolution of the Riemann–Hilbert problem and on isomonodromic deformations of first-order Fuchsian systems, which are given by the Schlesinger system.

**Theorem 2.** *Let  $P \subset \mathbb{R}^3$  be a closed polygon in generic position, with vertices  $A_1, \dots, A_{N+3}$  ( $N > 0$ ). Then there is some immersed minimal surface of disc-type which is bounded by  $P$  and maps the three points  $-1, -i, 1$  onto the last three vertices  $A_{N+1}, A_{N+2}, A_{N+3}$  of  $P$ .*

Here by *generic position*, we mean that the  $N + 3$  edge directions of the polygon  $P$  should satisfy the two following conditions: any two directions may not be parallel, and any three directions may not be coplanar.

In this paper, we have hence to show that one can carry over the above existence statement from such polygonal boundary curves to piecewise  $C^2$ –smooth closed Jordan curves  $\Gamma \subset \mathbb{R}^3$  with total curvature smaller than  $6\pi$ , by appropriately approximating such a curve  $\Gamma$  by polygons  $P^n$  in generic positions. We should note here that in [5] Garnier tried to use such a polygonal approximation process as well in order to obtain for any rectifiable, piecewise  $C^2$ –smooth, closed Jordan curve  $\Gamma \subset \mathbb{R}^3$ , which is merely assumed to have *finite* total curvature, the existence of some minimal surface  $X^* \in \mathcal{C}(\Gamma)$  as a limit of (indeed immersed) minimal surfaces  $X^n \in \mathcal{C}(P^n)$  with uniformly converging Weierstrass data  $(G^n, H^n)$ . Unfortunately, Garnier could only prove

that the Weierstrass data  $(G^n, H^n)$  of the minimal surfaces  $X^n \in \mathcal{C}(P^n)$  converge in  $C_{loc}^0(B)$ , i.e. uniformly on every interior subdomain  $B' \subset\subset B$ . Consequently he failed to explain in [5], pp. 116–144, why this limit surface  $X^*$  should map  $\partial B$  continuously and weakly monotonically onto  $\Gamma$  with mapping degree 1, i.e. why  $X^*$  should indeed be bounded by the approximated curve  $\Gamma$ . This could be one of the reasons for the fact that his paper [5] has never been accepted as the first rigorous solution of Plateau’s problem for arbitrary piecewise smooth boundary curves. Moreover the limit surface  $X^*$  of some sequence of minimal immersions  $X^n \in \mathcal{C}(P^n)$  whose Weierstrass data converge merely in  $C_{loc}^0(B)$  might in general have branch points in  $B$ . In fact,  $X^*$  could even be a constant map, thus a completely degenerated surface, if there is no further knowledge available neither about the involved boundary curves  $\Gamma$  and  $P^n$ , nor about the converging minimal immersions  $X^n$ .

In this paper, we at first reparametrize the minimal immersions  $X^n \in \mathcal{C}(P^n)$  appropriately in order to obtain the existence of some subsequence  $\{\tilde{X}^{n_i}\}$  whose boundary values can be shown to converge uniformly to a continuous and weakly monotonic parametrization  $\beta$  of the curve  $\Gamma$ . We then conclude in a second step that the harmonic extension  $X^*$  of  $\beta$  onto  $\bar{B}$  does not only inherit the conformality but also the absence of branch points from the uniformly converging minimal immersions  $\tilde{X}^{n_i}$ , by means of two Theorems due to Sauvigny [13], [14].

In contrast to this approximative strategy Osserman, Gulliver and Alt considered some global minimizer  $X^*$  of the Dirichlet energy  $\mathcal{D}(X) = \frac{1}{2} \int_B |DX|^2 \, dudv$  in the class  $\mathcal{C}(\Gamma)$  and classified the branch points of  $X^*$  into two different types, into “true” and “false” branch points. Alt showed firstly in [1] by a certain surgery technique that any “true” branch point  $w_0 \in B$  of  $X^*$  would lead to an impossible behaviour of the unit normal of  $X^*$  in a neighbourhood of  $w_0$ . After that he used in [2] another surgery technique in order to even rule out the existence of eventually remaining “false” branch points of any such global minimizer  $X^*$  of  $\mathcal{D}$  (in fact of any functional of some class which contains  $\mathcal{D}$  in particular). Since both of his exclusion arguments work for global minimizers in  $\mathcal{C}(\Gamma)$  for any closed rectifiable Jordan curve in  $\mathbb{R}^3$ , his result is obviously stronger than the above Theorem 1, to be proved in Section 3.

## 2 Preparations for the proof of Theorem 1

Firstly we need the following definitions.

**Definition 1.** Let  $\Gamma$  be an arbitrary closed rectifiable Jordan curve in  $\mathbb{R}^3$  being piecewise of class  $C^2$  and  $\gamma : \mathbb{S}^1 \xrightarrow{\cong} \Gamma$  a fixed parametrization of  $\Gamma$ .

- (i) We term the elements of a sequence  $\{P^n\}$  of simple closed polygons  $P^n \subset \mathbb{R}^3$  in generic positions *polygonal approximations* of  $\Gamma$  if there

exist homeomorphisms  $\varphi^n : \Gamma \xrightarrow{\cong} P^n$  such that

$$\max_{x \in \Gamma} |x - \varphi^n(x)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (1)$$

- (ii) We define the total curvature  $\text{TC}(P)$  of some polygonal approximation  $P$  as the sum of the exterior angles  $\eta_1, \dots, \eta_M$  of  $P$  at its consecutive vertices  $A_1, \dots, A_M$ .
- (iii) Finally, let  $L := \mathcal{L}(\Gamma)$  denote the length of  $\Gamma$  and consider the piecewise smooth parametrization  $\gamma : [0, L) \xrightarrow{\cong} \Gamma$  of  $\Gamma$  with  $\gamma \in C^2([0, L) \setminus \{t_1, \dots, t_m\})$  and  $|\dot{\gamma}| \equiv 1$  on  $[0, L) \setminus \{t_1, \dots, t_m\}$ , for some subdivision  $0 \leq t_1 < t_2 < \dots < t_m < L$  of  $[0, L)$ . We term  $\vartheta_i \in (0, \pi)$  the smaller angle between the two tangent vectors  $\dot{\gamma}(t_i+)$  and  $\dot{\gamma}(t_i-)$  and define the total curvature of  $\Gamma$  as

$$\text{TC}(\Gamma) := \sum_{i=1}^m \vartheta_i + \sum_{i=1}^m \int_{t_i}^{t_{i+1}} |\ddot{\gamma}| \, ds,$$

where we set  $t_{m+1} := t_1$ .

Now we can state the following technical approximation tool:

**Proposition 1.** *Let  $\Gamma$  be an arbitrary closed rectifiable Jordan curve in  $\mathbb{R}^3$  being piecewise of class  $C^2$  with three fixed different points  $x_0, x_1, x_2$ , and let  $\varepsilon > 0$  be arbitrarily fixed. Then there exist some sequence  $\{P^n\}$  of polygonal approximations of  $\Gamma$  with homeomorphisms  $\varphi^n : \Gamma \xrightarrow{\cong} P^n$  and some integer  $N(\varepsilon) \in \mathbb{N}$  such that there hold*

$$\mathcal{L}(P^n) < \mathcal{L}(\Gamma) + \varepsilon, \quad \text{and} \quad (2)$$

$$\text{TC}(P^n) < \text{TC}(\Gamma) + \varepsilon, \quad (3)$$

for every  $n > N(\varepsilon)$ . In addition, the sequence  $\{P^n\}$  can be constructed in such a way that three vertices  $a_0^n, a_1^n, a_2^n$  of each polygonal approximation  $P^n$  coincide with the fixed points  $x_0, x_1, x_2$  of  $\Gamma$ , i.e. such that  $a_0^n = x_0, a_1^n = x_1$  and  $a_2^n = x_2$  for each  $n \in \mathbb{N}$ , without violating the property to be in generic position of each  $P^n$ . Moreover, the homeomorphisms  $\varphi^n$  can then be chosen such that  $\varphi^n(x_k) = x_k$  ( $k = 0, 1, 2$ ) for every  $n \in \mathbb{N}$ .

Moreover, in the next section we will use the following two notions of *branch points* of some minimal surface with arbitrary continuous boundary values:

**Definition 2.** Let  $X \in C^0(\bar{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3)$  be an arbitrary minimal surface.

- (i) We call a point  $w_0 \in B$  an *interior branch point* of  $X$  if  $|X_u(w_0)| = 0$ .

(ii) We call a point  $w_0 \in \partial B$  a *boundary branch point* of  $X$  if there holds

$$|X_u(w_k)| \longrightarrow 0 \quad \text{as } k \rightarrow \infty \quad (4)$$

for any sequence  $\{w_k\} \subset B$  converging to  $w_0$ .

For any minimal surface  $X \in \mathcal{C}(P)$  bounded by some closed polygon  $P$ , Heinz proved in [7] in particular the following asymptotic expansion for the complex derivative  $X_w := \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v})X$  about the boundary points  $e^{it_1}, \dots, e^{it_M}$  that are mapped by  $X$  onto the vertices  $A_1, \dots, A_M$  of  $P$ :

$$X_w(w) = c_k (w - e^{it_k})^{\rho_k + m_k} + O(|w - e^{it_k}|^{\rho_k + m_k + \epsilon_k}), \quad (5)$$

for each  $k \in \{1, \dots, M\}$ . Here, the  $c_k$  are fixed vectors in  $\mathbb{C}^3 \setminus \{0\}$ , the  $m_k$  are non-negative integers – the so-called *branch point orders* of  $X$  with respect to the points  $e^{it_k}$  (see Definition 3 (i) below) –,  $\rho_k \in (-1, 0)$  and  $\epsilon_k \in (0, 1)$ . The angle of the minimal surface at the vertex  $A_k$  is  $(\rho_k + m_k + 1)\pi$ . If the integer  $m_k$  is even (resp. odd), then  $|\rho_k|\pi$  is the exterior angle  $\eta_k$  (resp. the interior angle) of the polygon  $P$  at the vertex  $A_k$ . Now one can easily see that such a point  $e^{it_k}$  is a *boundary branch point* of  $X$  in the sense of Definition 2 if and only if its order  $m_k$  is a *positive* integer.

Moreover, about any point  $w_0 \in \bar{B} \setminus \{e^{it_k}\}_{k=1}^M$  there is a Taylor expansion of  $X_w$ , i.e.

$$X_w(w) = a_{m(w_0)} (w - w_0)^{m(w_0)} + a_{m(w_0)+1} (w - w_0)^{m(w_0)+1} + \dots, \quad (6)$$

where  $a_{m(w_0)} \in \mathbb{C}^3 \setminus \{0\}$  and  $m(w_0) \in \mathbb{N}_0$ . Thus again, any point  $w_0 \in \bar{B} \setminus \{e^{it_k}\}_{k=1}^M$  is a branch point of the minimal surface  $X$  – in the sense of Definition 2 – if and only if the integer  $m(w_0)$  in (6) is positive. This motivates the following

**Definition 3.** (i) We term the exponent  $m_k$  in (5) resp.  $m(w_0)$  in (6) the branch point order of the surface  $X$  at the point  $e^{it_k}$ ,  $k = 1, \dots, M$ , resp. at the point  $w_0 \in \bar{B} \setminus \{e^{it_k}\}_{k=1}^M$ .

(ii) We define the total branch point order of  $X$  by

$$\kappa(X) := \sum_{w \in B} m(w) + \frac{1}{2} \sum_{w \in \partial B \setminus \{e^{it_k}\}_{k=1}^M} m(w) + \frac{1}{2} \sum_{k=1}^M m_k, \quad (7)$$

which is a finite sum since  $X$  can only have isolated and thus finitely many branch points on  $\bar{B}$  on account of the expansions (5) and (6).

The geometric meaning of the exponents  $\rho_k$ ,  $k=1, \dots, M$ , appearing in (5), becomes even clearer by means of the following Gauss-Bonnet formula for minimal surfaces with polygonal boundaries, the main result of Sauvigny's article [13]:

**Proposition 2.** *Let  $X \in \mathcal{C}(P)$  be a minimal surface bounded by some closed polygon  $P$ , with vertices  $A_1, \dots, A_M$ , having Gauss-curvature  $K_X$ . Then there holds the formula*

$$\int_B |K_X| E \, dudv + 2\pi(1 + \kappa(X)) = \pi \sum_{k=1}^M |\rho_k|, \quad (8)$$

where  $E$  denotes  $|X_u|^2 \equiv |X_v|^2$ .

For later purpose we should state the following

**Corollary 1.** *Let  $X \in \mathcal{C}(P)$  be a minimal surface bounded by some closed polygon  $P$  which has only branch points in the points  $e^{it_k}$ ,  $k = 1, \dots, M$ . Then there holds:*

$$\int_B |K_X| E \, dudv + 2\pi = \pi \sum_{k=1}^M (|\rho_k| - m_k). \quad (9)$$

Moreover, letting  $\eta_k$  denote the  $M$  exterior angles of  $P$ ,  $s$  the number of branch points of  $X$  and  $k_1, k_2, \dots, k_{M-s}$  those indices in  $\{1, \dots, M\}$  for which  $e^{it_{k_j}}$  is not a branch point of  $X$ , then we obtain the estimate

$$\int_B |K_X| E \, dudv \leq \sum_{j=1}^{M-s} \eta_{k_j} - 2\pi \quad (10)$$

and in particular

$$\int_B |K_X| E \, dudv \leq \text{TC}(P) - 2\pi, \quad (11)$$

where  $\text{TC}(P) := \sum_{k=1}^M \eta_k$  is the total curvature of  $P$ .

*Proof.* Formula (9) follows immediately from formula (8) and the definition of the total branch point order  $\kappa(X)$  in (7). Now, as explained above,  $\pi|\rho_k|$  measures exactly the exterior angle  $\eta_k$  of the polygon  $P$  at its vertex  $A_k$  if  $m_k$  is even, in particular if  $m_k = 0$ , i.e. if the point  $e^{it_k}$  is not a branch point of  $X$ . This yields  $\pi(|\rho_{k_j}| - m_{k_j}) = \eta_{k_j}$  for  $j = 1, \dots, M - s$  on the one hand. On the other hand, if some point  $e^{it_k}$  is a branch point of  $X$ , then we have  $m_k \geq 1$  and thus  $|\rho_k| - m_k \leq |\rho_k| - 1 < 0$  on account of  $\rho_k \in (-1, 0)$ . In combination with formula (9) this yields the claimed estimate (10) which instantly implies estimate (11) as well.  $\square$

Finally we need the following compactness result for boundary values, which we shall prove for the sake of completeness (see also [9], Paragraphs 21, 234 and 235):

**Proposition 3.** *Let  $\Gamma$  and  $\{P^n\}$  be as in Proposition 1 and  $X^n \in \mathcal{C}(P^n)$  some sequence of surfaces with uniformly bounded Dirichlet energies, i.e.  $\mathcal{D}(X^n) \leq M$  for every  $n \in \mathbb{N}$ , and satisfying a uniform three-point-condition  $X^n(e^{i\frac{\pi}{2}(2+k)}) = x_k$ , for  $k = 0, 1, 2$ , where  $x_0, x_1, x_2$  are three fixed consecutive points on  $\Gamma$ . Then there exists some subsequence  $\{X^{n_l}\}$  whose boundary values satisfy*

$$X^{n_l}|_{\partial B} \longrightarrow \beta \quad \text{in } C^0(\partial B, \mathbb{R}^3),$$

where  $\beta : \mathbb{S}^1 \longrightarrow \Gamma$  is a continuous, weakly monotonic map onto  $\Gamma$  with mapping degree 1 and  $\beta(e^{i\frac{\pi}{2}(2+k)}) = x_k$ , for  $k = 0, 1, 2$ .

*Proof.* We consider a fixed homeomorphic parametrization  $\gamma : \mathbb{S}^1 \xrightarrow{\cong} \Gamma$  of  $\Gamma$  and the weakly monotonic maps  $(\varphi^n)^{-1} \circ X^n|_{\partial B} : \partial B \longrightarrow \Gamma$  onto  $\Gamma$ . For each  $n \in \mathbb{N}$  there exist non-decreasing maps  $\sigma^n : [0, 2\pi] \longrightarrow [0, 4\pi]$ , with  $\sigma^n(2\pi) = \sigma^n(0) + 2\pi$ , such that  $(\varphi^n)^{-1} \circ X^n(e^{it}) = \gamma(e^{i\sigma^n(t)})$  for all  $t \in [0, 2\pi]$ . By (1) we conclude that

$$\begin{aligned} \max_{t \in [0, 2\pi]} \left| \gamma(e^{i\sigma^n(t)}) - X^n(e^{it}) \right| &= \max_{t \in [0, 2\pi]} \left| \gamma(e^{i\sigma^n(t)}) - \varphi^n(\gamma(e^{i\sigma^n(t)})) \right| \\ &= \max_{x \in \Gamma} |x - \varphi^n(x)| \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{12}$$

Furthermore, Helly's selection principle (see [8], p. 248) yields some subsequence  $\{\sigma^{n_l}\}$  and a non-decreasing function  $\sigma$  on  $[0, 2\pi]$  such that

$$\sigma^{n_l}(t) \longrightarrow \sigma(t) \quad \forall t \in [0, 2\pi], \quad \text{as } l \rightarrow \infty, \tag{13}$$

thus also  $\gamma(e^{i\sigma^{n_l}(t)}) \longrightarrow \gamma(e^{i\sigma(t)})$  for all  $t \in [0, 2\pi]$ . Hence, together with (12) we arrive at

$$X^{n_l}(e^{it}) \longrightarrow \gamma(e^{i\sigma(t)}) \quad \forall t \in [0, 2\pi], \quad \text{as } l \rightarrow \infty, \tag{14}$$

which especially implies  $\gamma(e^{i\sigma(\frac{\pi}{2}(2+k))}) = x_k$ , for  $k = 0, 1, 2$ , due to the required uniform three-point-condition imposed on  $X^n|_{\partial B}$ , for each  $n$ . Hence, since  $x_i \neq x_j$  for  $i \neq j$  we see that

$$\sigma\left(\frac{\pi}{2}(2+i)\right) \neq \sigma\left(\frac{\pi}{2}(2+j)\right) \quad \text{mod } 2\pi, \quad \text{for } i \neq j. \tag{15}$$

Now an extension of Helly's selection principle (see [11], p. 63 and p. 226) provides the uniform convergence of the  $\sigma^{n_l}$  if  $\sigma$  is known to be continuous, what we are going to prove now. To this end we shall assume that  $\sigma$  was not continuous. Since  $\sigma$  is weakly monotonic, there exist the one-sided limits  $\sigma(t+0)$  and  $\sigma(t-0)$ , for all  $t \in [0, 2\pi]$ , where we mean  $\sigma(0-0) := \sigma(2\pi-0) - 2\pi$  and  $\sigma(2\pi+0) := \sigma(0+0) + 2\pi$ . The points of discontinuity of  $\sigma$  coincide with those points  $t^*$  in which we have  $0 < \sigma(t^*+0) - \sigma(t^*-0)$ .

Moreover there holds  $\sigma(t^*+0) - \sigma(t^*-0) < 2\pi$ , otherwise on account of the monotonicity of  $\sigma$  and  $\sigma(2\pi) = \sigma(0) + 2\pi$  we would have  $\sigma(t) \equiv \sigma(t^*-0)$  on  $[0, t^*)$  and  $\sigma(t) \equiv \sigma(t^*+0)$  on  $(t^*, 2\pi]$ , which contradicts (15). Hence, we conclude that  $\sigma(t^*+0) \neq \sigma(t^*-0) \pmod{2\pi}$  and therefore by the injectivity of  $\gamma$

$$\gamma\left(e^{i\sigma(t^*+0)}\right) \neq \gamma\left(e^{i\sigma(t^*-0)}\right) \quad (16)$$

in every discontinuity point  $t^*$  of  $\sigma$ . Now we fix such a point  $t^*$  which we may suppose to be contained in  $(0, 2\pi)$  without loss of generality. By (16) we have  $|\gamma(e^{i\sigma(t^*+0)}) - \gamma(e^{i\sigma(t^*-0)})| = \epsilon > 0$  for some  $\epsilon > 0$ . Moreover by the existence of the one-sided limits  $\sigma(t^*+0)$ ,  $\sigma(t^*-0)$  and by the continuity of  $\gamma$  there is some sufficiently small  $\alpha > 0$  such that  $[t^* - \alpha, t^* + \alpha] \subset (0, 2\pi)$  and

$$\begin{aligned} & \left| \gamma\left(e^{i\sigma(t)}\right) - \gamma\left(e^{i\sigma(t^*-0)}\right) \right| < \frac{\epsilon}{3} \quad \forall t \in (t^* - \alpha, t^*) \\ \text{and} \quad & \left| \gamma\left(e^{i\sigma(t)}\right) - \gamma\left(e^{i\sigma(t^*+0)}\right) \right| < \frac{\epsilon}{3} \quad \forall t \in (t^*, t^* + \alpha), \end{aligned}$$

which implies together with (14):

$$\lim_{l \rightarrow \infty} \left| X^{n_l}\left(e^{it'}\right) - X^{n_l}\left(e^{it''}\right) \right| = \left| \gamma\left(e^{i\sigma(t')}\right) - \gamma\left(e^{i\sigma(t'')}\right) \right| > \frac{\epsilon}{3} \quad (17)$$

for all  $t' \in (t^* - \alpha, t^*)$  and all  $t'' \in (t^*, t^* + \alpha)$ . Now we only consider pairs  $t', t''$  such that  $0 < t'' - t^* = t^* - t' < \alpha$ . For  $r := 2 \sin\left(\frac{t^* - t'}{2}\right)$  we have  $\partial B_r(e^{it^*}) \cap \partial B = \{e^{it'}, e^{it''}\}$ . We introduce the notation  $\{w_1(\rho), w_2(\rho)\} := \partial B_\rho(e^{it^*}) \cap \partial B$ , for  $0 < \rho < 2 \sin\left(\frac{\alpha}{2}\right)$ . Now making use of the requirement  $\mathcal{D}(X^n) \leq M$  for all  $n \in \mathbb{N}$ , and of Hölder's inequality one easily infers from Fatou's lemma that  $\liminf_{l \rightarrow \infty} \left| X^{n_l}(w_1(\rho)) - X^{n_l}(w_2(\rho)) \right|^2 \frac{1}{\rho} \in L^1([\delta, \sqrt{\delta}])$  for any  $\delta < 4 \sin^2\left(\frac{\alpha}{2}\right)$  and that there holds (see [9], pp. 207–209):

$$\frac{1}{2\pi} \int_{\delta}^{\sqrt{\delta}} \liminf_{l \rightarrow \infty} \left| X^{n_l}(w_1(\rho)) - X^{n_l}(w_2(\rho)) \right|^2 \frac{d\rho}{\rho} \leq M.$$

Combining this with (17) we achieve:

$$M > \frac{\epsilon^2}{18\pi} \int_{\delta}^{\sqrt{\delta}} \frac{d\rho}{\rho} = \frac{\epsilon^2}{36\pi} \log\left(\frac{1}{\delta}\right) \quad \forall \delta < 4 \sin^2\left(\frac{\alpha}{2}\right),$$

which yields a contradiction for a sufficiently small choice of  $\delta$ . Hence,  $\sigma$  must be continuous on  $[0, 2\pi]$  and therefore the convergence in (13) even uniform:

$$\sigma^{n_l} \longrightarrow \sigma \quad \text{in } C^0([0, 2\pi]).$$

As  $\gamma$  is uniformly continuous on  $\mathbb{S}^1$  this yields

$$\gamma\left(e^{i\sigma^{n_l}(\cdot)}\right) \longrightarrow \gamma\left(e^{i\sigma(\cdot)}\right) \quad \text{in } C^0([0, 2\pi], \mathbb{R}^3),$$



and together with (12) we finally arrive at

$$X^{n_i} \left( e^{i(\cdot)} \right) \longrightarrow \gamma \left( e^{i\sigma(\cdot)} \right) \quad \text{in } C^0([0, 2\pi], \mathbb{R}^3). \quad (18)$$

Hence, defining  $\beta : \mathbb{S}^1 \longrightarrow \Gamma$  via  $\beta(e^{i(\cdot)}) := \gamma(e^{i\sigma(\cdot)})$  we see that  $\beta$  has in fact the asserted properties on account of the continuity and weak monotonicity of  $\sigma$  and of its property  $\sigma(2\pi) = \sigma(0) + 2\pi$ , in combination with the homeomorphy of  $\gamma$ . Finally  $\beta(e^{i\frac{\pi}{2}(2+k)}) = x_k$ , for  $k = 0, 1, 2$ , follows immediately from (18).  $\square$

### 3 Proof of Theorem 1

Now we fix some closed rectifiable, piecewise  $C^2$ -Jordan curve  $\Gamma$  and choose three different consecutive points  $x_0, x_1, x_2$  on  $\Gamma$ . By Proposition 1 we obtain some sequence  $\{P^n\}$  of polygonal approximations of  $\Gamma$  with  $N_n + 3$  vertices, which we shall enumerate in the following manner:

$$(a_0^n, A_1^n, \dots, A_{l_n}^n; a_1^n; A_{l_n+1}^n, \dots, A_{m_n}^n; a_2^n; A_{m_n+1}^n, \dots, A_{N_n}^n), \quad (19)$$

such that  $a_0^n = x_0, a_1^n = x_1$  and  $a_2^n = x_2$  for each  $n \in \mathbb{N}$ . Now, since each polygonal approximation  $P^n$  is in generic position, Theorem 2 guarantees the existence for each  $n$  of some immersed minimal surface  $X^n$  spanning  $P^n$ , and mapping the three points  $-1, -i, 1$  onto the last three vertices in (19). To establish Theorem 1, we are now going to successively apply Proposition 3 and a theorem due to Sauvigny (Theorem 1, (ii) in [14]) to a sequence  $\{\tilde{X}^n\}$  of immersions obtained from  $\{X^n\}$  by an appropriate reparametrization.

Since the boundary values of the surfaces  $X^n$  are known to map  $\partial B$  (weakly) monotonically onto  $P^n$  with mapping degree 1 we can estimate the Dirichlet energies, resp. the areas, of the  $X^n$  by means of the isoperimetric inequality and (2):

$$\mathcal{D}(X^n) = \mathcal{A}(X^n) \leq \frac{1}{4\pi} \text{Tot.Var.}(X^n|_{\partial B})^2 = \frac{1}{4\pi} \mathcal{L}(P^n)^2 \leq c \mathcal{L}(\Gamma)^2, \quad (20)$$

for each  $n \in \mathbb{N}$ , where  $c$  is a positive constant. Moreover there is a unique biholomorphic automorphism  $\Phi^n : B \xrightarrow{\cong} B$ , with a unique homeomorphic extension onto  $\bar{B}$ , such that the reparametrization  $\tilde{X}^n := X^n \circ \Phi^n$  maps the three points  $-1, -i, 1$  onto the three specified vertices  $a_0^n = x_0, a_1^n = x_1$  and  $a_2^n = x_2$  of  $P^n$ , which are fixed on  $\Gamma$  as  $n \rightarrow \infty$ . As the automorphism  $\Phi^n$  is biholomorphic on  $B$  and as its extension  $\Phi^n|_{\partial B} : \partial B \xrightarrow{\cong} \partial B$  performs an orientation preserving homeomorphism, the reparametrized surface  $\tilde{X}^n$  is again an immersed minimal surface spanning  $P^n$  which in addition meets the *uniform three-point-condition* of Proposition 3. And due to  $\mathcal{D}(\tilde{X}^n) = \mathcal{D}(X^n)$  the surface  $\tilde{X}^n$  clearly also satisfies the estimate (20). Thus we can apply

Proposition 3 in order to obtain the existence of some subsequence  $\tilde{X}^{n_l}$  whose boundary values  $\tilde{X}^{n_l}|_{\partial B}$  satisfy

$$\tilde{X}^{n_l}|_{\partial B} \longrightarrow \beta \quad \text{in } C^0(\partial B, \mathbb{R}^3),$$

where  $\beta : \mathbb{S}^1 \longrightarrow \Gamma$  is a continuous, weakly monotonic map onto  $\Gamma$  with mapping degree 1. Now, by the maximum principle and Cauchy's estimates we can immediately conclude that the subsequence  $\tilde{X}^{n_l}$  converges in  $C^0(\bar{B}) \cap C_{loc}^2(B)$  to the unique harmonic extension  $X^*$  of  $\beta$  onto  $\bar{B}$  which is thus again conformally parametrized and bounded by  $\Gamma$ .

Now thanks to Theorem 1, (ii) in [14], to prove that the harmonic extension  $X^*$  is free of interior branch points, it is sufficient to show that there is a constant  $e_0 \in (0, 4\pi)$  such that there holds for every sufficiently large  $l$ :

$$\int_B |K_{\tilde{X}^{n_l}}| \tilde{E}^{n_l} du dv \leq e_0$$

with  $\tilde{E}^{n_l} := |\tilde{X}_u^{n_l}|^2 \equiv |\tilde{X}_v^{n_l}|^2$ . We fix some  $n$  arbitrarily. We know by Theorem 2 that  $X^n$  maps some  $N_n$ -tuple of points  $e^{it_1^n}, e^{it_2^n}, \dots, e^{it_{N_n}^n} \in \mathbb{S}^1 \cap \{\Im(w) > 0\}$ , precisely with  $0 < t_1^n < t_2^n < \dots < t_{N_n}^n < \pi$ , onto the first  $N_n$  vertices of  $P^n$  in (19) and the three points  $-1, -i, 1$  onto the last three vertices of  $P^n$ . The set of branch points  $\mathcal{B}^n$  of  $X^n$ , which is a subset of the set  $\mathcal{S}^n := \{e^{it_1^n}, e^{it_2^n}, \dots, e^{it_{N_n}^n}, -1, -i, 1\}$ , consists of  $s^n \in \{0, 1, \dots, N_n + 3\}$  points. For simplification of notation and without loss of generality, we shall suppose that these  $s^n$  branch points of  $X^n$  are adjacent, e.g.  $\mathcal{B}^n = \{e^{it_1^n}, e^{it_2^n}, \dots, e^{it_{s^n}^n}\}$ . We shall term the  $N_n + 3$  exterior angles in the consecutive vertices (19) of  $P^n$ :  $\eta_1^n, \eta_2^n, \dots, \eta_{N_n+3}^n$ . Now, applying estimate (10) to each surface  $X^n$  with branch point set  $\mathcal{B}^n = \{e^{it_1^n}, e^{it_2^n}, \dots, e^{it_{s^n}^n}\}$  we obtain:

$$\int_B |K_{X^n}| E^n du dv \leq \sum_{j=s^n+1}^{N_n+3} \eta_j^n - 2\pi, \quad (21)$$

$\forall n \in \mathbb{N}$ . Thus, applying the weaker estimate (11) to each surface  $X^n$  in combination with the requirement that  $\text{TC}(\Gamma) = 6\pi - 2\varepsilon$ , for some  $\varepsilon > 0$ , we achieve by (3) in Proposition 1 the existence of some large integer  $N(\varepsilon)$  such that there holds

$$\int_B |K_{X^n}| E^n du dv \leq \text{TC}(P^n) - 2\pi < \text{TC}(\Gamma) + \varepsilon - 2\pi = 4\pi - \varepsilon, \quad (22)$$

for the integral over the (negative) Gaussian curvature  $K_{X^n}$  of  $X^n$ , whenever  $n > N(\varepsilon)$ . Finally, since there holds  $K_{\tilde{X}^n} = K_{X^n} \circ \Phi^n$  on  $B$  for the Gaussian curvature of  $\tilde{X}^n$  on account of the Theorema Egregium and the biholomorphy

of  $\Phi^n$  we deduce from (22) that the reparametrized minimal surface  $\tilde{X}^n$  again satisfies the estimate

$$\begin{aligned} \int_B |K_{\tilde{X}^n}| \tilde{E}^n \, dudv &= \int_B |K_{X^n}| \circ \Phi^n E^n \circ \Phi^n |(\Phi^n)'|^2 \, dudv \\ &\equiv \int_B |K_{X^n}| \circ \Phi^n E^n \circ \Phi^n \det(D_{(u,v)}\Phi^n) \, dudv \\ &= \int_B |K_{X^n}| E^n \, dudv < 4\pi - \varepsilon, \end{aligned}$$

whenever  $n > N(\varepsilon)$ . Thus by Theorem 1, (ii) in [14] due to Sauvigny we may infer that the limit surface  $X^*$  in fact inherits the absence of interior branch points of the converging minimal surfaces  $\tilde{X}^{n_i}$ , i.e. that  $X^*$  is free of branch points on the open unit disc  $B$ , just as asserted in Theorem 1.

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## References

- [1] Alt, H.W.: Verzweigungspunkte von H-Flächen I. Math. Z. **127**, 333–362 (1972).
- [2] Alt, H.W.: Verzweigungspunkte von H-Flächen II. Math. Ann. **201**, 33–55 (1973).
- [3] Desideri, L.: Problème de Plateau, équations fuchsiennes et problème de Riemann–Hilbert (*The Plateau problem, Fuchsian equations and the Riemann–Hilbert problem*). To appear in Mémoires de la Soc. Math. Fr., arXiv: 1003.0978.
- [4] Douglas, J.: Solutions of the problem of Plateau. Trans. Amer. Math. Soc. **33**, 263–321 (1931).
- [5] Garnier, R.: Le problème de Plateau. Annales scientifiques de l'É.N.S. **45**, 53–144 (1928).
- [6] Gulliver, R.D.: Regularity of minimizing surfaces of prescribed mean curvature. Ann. of Math. (2) **97**, 275–305 (1973).
- [7] Heinz, E.: Über die analytische Abhängigkeit der Lösungen eines linearen elliptischen Randwertproblems von den Parametern. Nachr. Akad. Wiss. Gött., Math.-Phys. Kl.II., 1–20 (1979).
- [8] Natanson, I.P.: Theorie der Funktionen einer reellen Veränderlichen. Akademie-Verlag, Berlin, 1969.

- [9] Nitsche, J.C.C.: Vorlesungen über Minimalflächen. Grundlehren der mathematischen Wissenschaften **199**, Springer-Verlag, Berlin, 1975.
- [10] Osserman, R.: A proof of the regularity everywhere of the classical solution to Plateau's problem. *Ann. of Math. (2)* **91**, 550–569 (1970).
- [11] Pólya, G., Szegő, G.: Aufgaben und Lehrsätze aus der Analysis. 3rd edition, Vol. I, Springer-Verlag, Berlin, 1964.
- [12] Rado, T.: On Plateau's problem. *Ann. of Math. (2)* **31**, 457–469 (1930).
- [13] Sauvigny, F.: On the total number of branch points of quasi-minimal surfaces bounded by a polygon. *Analysis* **8**, 297–304 (1988).
- [14] Sauvigny, F.: On immersions of constant mean curvature: Compactness results and finiteness theorems for Plateau's Problem. *Arch. Rat. Mech. Anal.* **110**, 125–140 (1990).